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CONFORMAL MAPPING FOR HEAT CONDUCTION IN A REGION WITH AN UNKNOWN BOUNDARY

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INTRODUCTION

FOR HEAT conduction in a fixed region the specification of the local surface temperature or local heat flux normal to the surface is sufficient to determine the temperature distribution within the region. The use of conformal mapping for this type of heat conduction problem in two dimensions has been discussed in [1]. If the surface temperature and heat flux are both specified the shape of the region must be free to adjust to accommodate both of these conditions. This note deals with the application of conformal mapping to two-dimensional heat conduction problems where the shape of the conducting region is unknown and will either adjust itself or is to be shaped in order to satisfy the imposed thermal conditions.

The method is best illustrated by considering a specific example. Thus, consider the geometry shown in Fig. 1. A cooled surface maintained at the temperature t_w is insulated at its sides, and the length normal to the plane of the figure is sufficiently long so that the geometry can be considered two dimensional. There is a region of conducting material on the plate. The upper surface of this region is isothermal and is subjected to a unidirectional source of thermal radiation. This region might be, for example, a steady state frost layer which has formed on a very cold plate exposed to the sun's rays. Since the frost surface is at the freezing or sublimation temperature consistent with the surrounding conditions, it will be at a constant temperature t_s. It is desired to find the shape that the frost region assumes and the heat flow through this region since this determines how well the frost layer insulates the surface. Alternatively the results can be interpreted as the solution to the problem of finding the shape

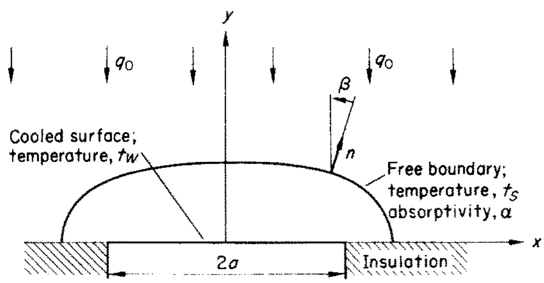

FIG. 1. Cross section of two-dimensional region with free boundary at uniform temperature t_s and with uniform absorptivity α exposed to unidirectional radiation q_0.

of a conducting region that will provide a uniform temperature t_s at its surface when this surface is subjected to incident radiation. It will be assumed that t_s is sufficiently low so that radiation emitted from the surface can be neglected compared with the absorbed incident radiation.

ANALYSIS

To begin the analysis, a dimensionless temperature T is defined as $T = (t - t_w)/(t_s - t_w)$ so that at the cooled plate $T = 0$, and at the irradiated free surface $T = 1$. Let α be the total absorptivity for the incident radiation flux q_0, and define a length scale parameter $\gamma = k(t_s - t_w)/\alpha q_0$ so that $X = x/\gamma$, $Y = y/\gamma$, etc. The dimensionless configuration is shown in Fig. 2.

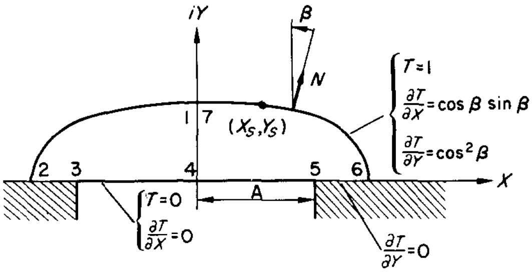

FIG. 2. Conducting region and boundary conditions in dimensionless physical plane, $Z = X + iY$.

At any position along the free boundary let the angle between the outward normal and the incident flux be β. Then the local heat conducted into the surface will be $\alpha q_0 \cos\beta = k\partial t/\partial n$. In dimensionless form this becomes at any position along the free boundary $\partial T/\partial N = \cos\beta$ where $-\pi/2 \leqslant \beta \leqslant \pi/2$. The symmetry of the problem implies that in particular $\partial T/\partial N = 1$ at $X = 0$. By resolving the derivative into components we obtain $\partial T/\partial X = \cos\beta\sin\beta$, $\partial T/\partial Y = \cos^2\beta$ for $-\pi/2 \leqslant \beta \leqslant \pi/2$.

Since $-T$ is a harmonic function of X and Y (i.e. $-T$ satisfies Laplace's equation) within the conducting region there exists a harmonic function ψ such that $W \equiv -T + i\psi$* is an analytic function of the complex variable $Z = X + iY$. (The lines ψ = constant are normal to the constant temperature lines and are therefore in the direction of heat flow). In addition the function ζ defined by

$$\zeta \equiv \frac{dW}{dZ} = -\frac{\partial T}{\partial X} + i\frac{\partial T}{\partial Y} \qquad (1)$$

is also an analytic function of the complex variable Z.

Now the boundary conditions shown in Fig. 2 are sufficient to determine the shape of the heat conducting region in both the complex W-plane (potential plane) and the complex ζ-plane (temperature derivative plane). In particular since there is no heat flow into the insulating material, the local heat flow is directed along the boundaries $\overline{23}$ and $\overline{56}$ in Fig. 2. These boundaries are therefore lines of constant ψ. The boundaries $\widehat{345}$ and $\widehat{2176}$ are specified as lines of constant T. Hence the conducting region must occupy the rectangular region in the potential plane shown in Fig. 3. The same numbers are used to designate corresponding points in the various planes.

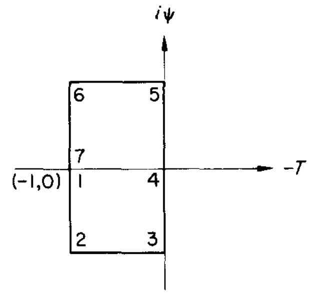

FIG. 3. Region mapped into potential plane, $W = -T + i\psi$.

In order to find the shape of the conducting region in the ζ-plane notice that on the free boundary $\widehat{2176}$

$$\left(\frac{\partial T}{\partial X}\right)^2 + \left(\frac{\partial T}{\partial Y}\right)^2 = \cos^2\beta\sin^2\beta + \cos^4\beta = \cos^2\beta = \frac{\partial T}{\partial Y}.$$

Hence

$$\left(\frac{\partial T}{\partial X}\right)^2 + \left(\frac{\partial T}{\partial Y} - \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2$$

which is the equation of a circle in the ζ-plane with radius $\frac{1}{2}$ and with its center at the point $(0, \frac{1}{2})$. Since along $\widehat{345}$ the temperature is constant and hence $\partial T/\partial X = 0$, the shape of the conducting region in the ζ-plane must be as shown in Fig. 4.

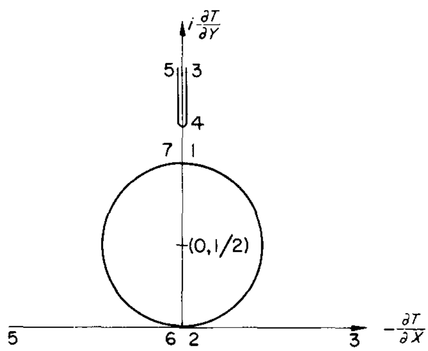

FIG. 4. Region mapped into temperature derivative plane, $\zeta = -\partial T/\partial X + i\,\partial T/\partial Y$.

Now integrating equation (1) yields

$$Z = \int_0^W \frac{1}{\zeta} dW. \qquad (2)$$

By conformally mapping the regions in the W and ζ planes into a common region in some intermediate plane, the two

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* The function $\psi + iT$ is also used in the literature which is the W used here multiplied by i.

functions W and ζ can be related. Then the integration indicated in equation (2) can be carried out to obtain the configuration of the conducting region in the physical plane.

The common region used to relate W and ζ is the uppei half u-plane shown in Fig. 5. The mapping function from

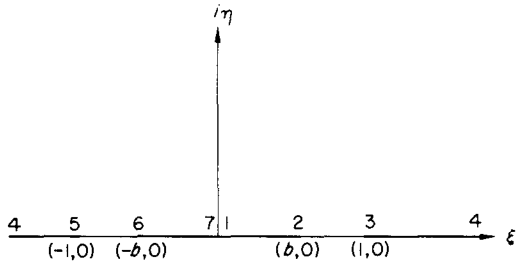


FIG. 5. Intermediate u plane, $u = \xi + i\eta$.

the W-plane to the u-plane is found upon application of the Schwarz–Christoffel transformation [2] to be

$$\frac{\mathrm{d}W}{\mathrm{d}u} = \frac{C_1}{\sqrt{(u^2 - b^2)}\sqrt{(u^2 - 1)}}. \tag{3}$$

To relate ζ to u a reciprocal transformation

$$\omega = \frac{1}{\zeta} \tag{4}$$

is used first to map the region in the ζ-plane into the region shown in Fig. 6. Then the Schwarz–Christoffel transformation [2] is used to map this region in the ω-plane into the

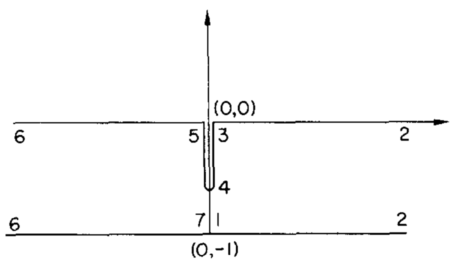


FIG. 6. Intermediate ω-plane.

common region in the u-plane. This transformation is given by

$$\frac{\mathrm{d}\omega}{\mathrm{d}u} = \frac{C_2}{(u^2 - b^2)\sqrt{(u^2 - 1)}}. \tag{5}$$

Equation (5) can be integrated and corresponding points matched in the ω and u planes to evaluate C_2 and the constant of integration. This yields

$$\omega = \frac{1}{\zeta} = -\frac{2}{\pi}\log\left(\frac{u\sqrt{(1 - b^2)} + ib\sqrt{(u^2 - 1)}}{\sqrt{(u^2 - b^2)}}\right). \tag{6}$$

The constant C_1 in equation (3) is evaluated by using the fact that $T_6 - T_5 = 1$. Thus,

$$-[W(6) - W(5)] = 1 = \int_{-1}^{-b} \frac{C_1\,\mathrm{d}\xi}{\sqrt{(\xi^2 - b^2)}\sqrt{(\xi^2 - 1)}} = \frac{C_1}{i} K[\sqrt{(1 - b^2)}]$$

so that

$$C_1 = \frac{i}{K[\sqrt{(1 - b^2)}]} \tag{7}$$

where K is the complete elliptic integral of the first kind.

By use of equations (2), (3), (6) and (7) the intermediate variable u can be related to the physical coordinates. Thus

$$Z = A + \int_{u=-1}^{u} \frac{1}{\zeta}\frac{\mathrm{d}W}{\mathrm{d}u}\mathrm{d}u = A - \frac{i2}{\pi K[\sqrt{(1 - b^2)}]} \int_{-1}^{u} \left[\log\left(\frac{\tilde{u}\sqrt{(1 - b^2)} + ib\sqrt{(\tilde{u}^2 - 1)}}{\sqrt{(\tilde{u}^2 - b^2)}}\right)\right] \frac{\mathrm{d}\tilde{u}}{\sqrt{(\tilde{u}^2 - b^2)}\sqrt{(\tilde{u}^2 - 1)}} \tag{8}$$

where $\tilde{u}$ is a dummy variable of integration.

In order to obtain an expression for the dimensionless length A which relates the imposed physical quantities to the mapping parameter b, evaluate the integral of equation (2), as in equation (8), between points 4 and 3 which correspond to the points $\xi = \infty$ and 1, respectively, on the real axis of the u-plane. Notice that the real part of the logarithm in equation (8) is zero on this portion of the real axis [2]. Hence this yields

$$A = \frac{2}{\pi K[\sqrt{(1 - b^2)}]} \int_{1}^{\infty} \tan^{-1}\left(\frac{b\sqrt{(\xi^2 - 1)}}{\xi\sqrt{(1 - b^2)}}\right) \frac{\mathrm{d}\xi}{\sqrt{(\xi^2 - b^2)}\sqrt{(\xi^2 - 1)}}$$

or after setting $v = 1/\xi$,

$$A = \frac{\alpha q_0 a}{k(t_s - t_w)} = \frac{2}{\pi K[\sqrt{(1 - b^2)}]} \int_{0}^{1} \tan^{-1}\left(\frac{b\sqrt{(1 - v^2)}}{\sqrt{(1 - b^2)}}\right) \frac{\mathrm{d}v}{\sqrt{(1 - v^2 b^2)}\sqrt{(1 - v^2)}}. \tag{9}$$

The free boundary which extends between points 6 and 7 maps onto the real axis of the u-plane. Its coordinates can therefore be found by integrating equation (8) along the real axis of the u-plane from $\xi = -1$ to values of ξ lying between $-b$ and 0. The part of the integral from $\xi = -1$

to $-b$ is evaluated as a separate constant. By taking the real and imaginary parts of the resulting expression the X and Y coordinates of the free boundary (denoted by X_s and Y_s respectively) are found to be

$$\frac{X_s}{A}=\frac{x}{a}=1+\frac{2}{\pi AK[\sqrt{(1-b^2)}]}\left\{\int_b^1 \log\left(\frac{b\sqrt{(1-\delta^2)}+\delta\sqrt{(1-b^2)}}{\sqrt{(\delta^2-b^2)}}\right)\right.$$

$$\left.\times\frac{d\delta}{\sqrt{(\delta^2-b^2)}\sqrt{(1-\delta^2)}}-\frac{\pi}{2}\left[F\left(\sin^{-1}\frac{\xi}{b},b\right)+K(b)\right]\right\} \quad (-b<\xi\leqslant 0) \qquad (10a)$$

$$\frac{Y_s}{A}=\frac{y}{a}=\frac{2}{\pi AK[\sqrt{(1-b^2)}]}$$

$$\times\int_{-b}^{\xi}\log\left(\frac{b\sqrt{(1-\bar{\xi}^2)}-\bar{\xi}\sqrt{(1-b^2)}}{\sqrt{(b^2-\bar{\xi}^2)}}\right)$$

$$\times\frac{d\bar{\xi}}{\sqrt{(b^2-\bar{\xi}^2)}\sqrt{(1-\bar{\xi}^2)}} \quad (-b<\xi\leqslant 0) \qquad (10b)$$

where $\bar{\xi}$ is a dummy variable of integration and F is the elliptic integral of the first kind [3].

The heat flow Q through the frozen region per unit length normal to the two-dimensional cross section is calculated as follows:

Notice that

$$Q=2\int_{x=0}^{a}k\left(\frac{\partial t}{\partial y}\right)_{y=0}dx=2k(t_s-t_w)\int_{X=0}^{A}\left(\frac{\partial T}{\partial Y}\right)_{Y=0}dX.$$

Hence by using the Cauchy Riemann relation $\partial T/\partial Y = \partial\psi/\partial X$ [2] the integration can be carried out to give

$$\frac{Q}{2k(t_s-t_w)}=\psi(5)-\psi(4)=\psi(6)-\psi(7)$$

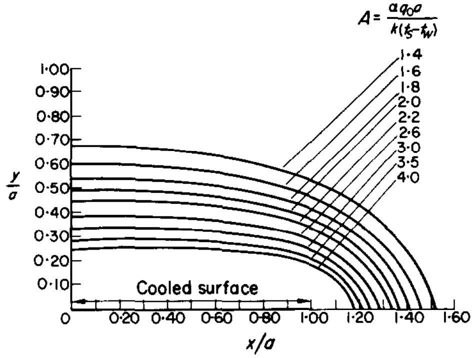
$$=Im\,[W(6)-W(7)]=Im\int_{u=0}^{-b}\frac{dW}{du}du.$$

Finally substituting equations (3) and (7) into the integral shows that

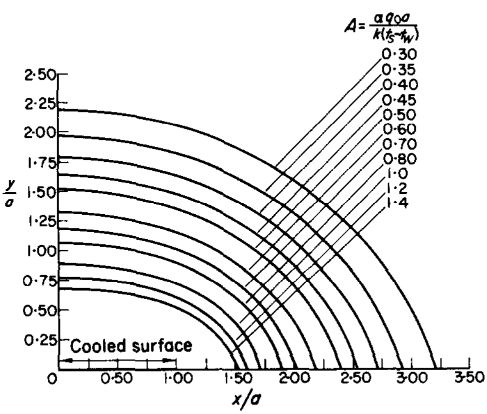
$$\frac{Q}{2k(t_s-t_w)}=\frac{K(b)}{K[\sqrt{(1-b^2)}]}. \qquad (11)$$

RESULTS

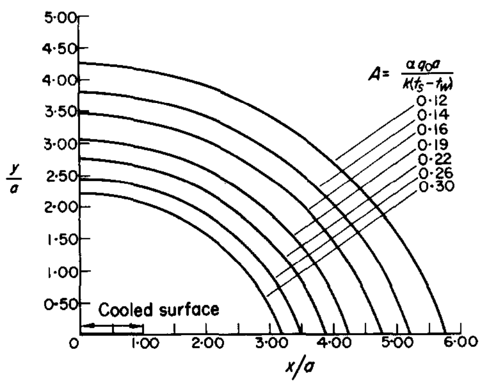
Equations (9)–(11) relate the imposed physical conditions (which group into the single parameter $A = \alpha q_0 a/k[t_s - t_w]$) to the coordinates of the free boundary and to the heat flow. These quantities are evaluated for various values of the mapping parameter b and can then be expressed in terms of each other.



(a) Thin regions, $1{\cdot}4 \leq A$



(b) Regions of intermediate thickness, $0{\cdot}3 \leq A \leq 1{\cdot}4$



(c) Thick regions, $0{\cdot}12 \leq A \leq 0{\cdot}3$

FIG. 7. Frozen region for various values of the physical parameter, $A = \alpha q_0 a/k(t_s - t_w)$.

(a) Thin regions, $1{\cdot}4 \leqslant A$.
(b) Regions of intermediate thickness, $0{\cdot}3 \leqslant A \leqslant 1.4$.
(c) Thick regions, $0{\cdot}12 \leqslant A \leqslant 0{\cdot}3$.

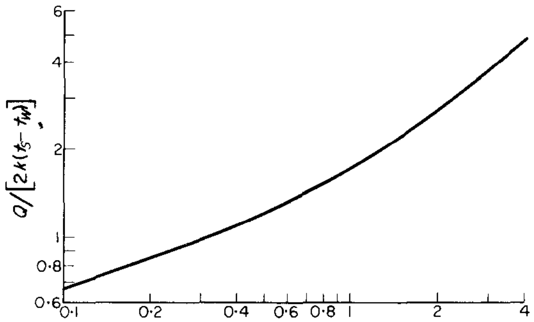


FIG. 8. Dimensionless heat flow through frozen region as a function of physical parameter involving absorbed incident radiation.

The results are given in Figs. 7 and 8 which show the configurations of the free boundary and the heat flow through the region as a function of the physical parameter A. The large values of A are associated with large values of the absorbed radiative heat flux, or small values of the cooling temperature below the surface temperature. These conditions yield thin conducting regions. The large A are also associated with large values of the dimensionless heat flow as shown in Fig. 8.

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APPLICATION OF A SAMPLED-DATA MODEL TO THE TRANSIENT RESPONSE OF A DISTRIBUTED PARAMETER SYSTEM SUBJECT TO SIMULTANEOUS RADIATION AND CONVECTION

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INTRODUCTION

A DISTRIBUTED parameter system represented by the one-dimensional heat conduction equation subject to both radiation and convection is studied. In the analysis the input function is assumed to be sampled and held and by employing the Laplace transform and the z-transforms respectively, a discrete-time system of equations is obtained for digital computer solution. Results obtained were in excellent agreement with published ones of Crosbie and Viskanta [1]. The method of solution is a direct one and no iteration techniques are required.

STATEMENT OF THE PROBLEM

We shall be concerned with the problem of obtaining the transient heating and cooling solutions for the one-dimensional slab initially at a uniform distribution and then subjected to both radiation and convection at one of its boundaries. The assumptions made and the nomenclature used are identical to those of Crosbie and Viskanta [1] and hence will not be repeated here. The basic system equation is then given by

$$\frac{\partial u}{\partial t}(x, t) = \frac{\partial^2 u}{\partial x^2}(x, t); \qquad \begin{matrix} t > 0 \\ 0 < x < 1 \end{matrix} \tag{1}$$

and the initial and boundary conditions are

$$u(x, 0) = u_i \tag{2a}$$

$$\frac{\partial u}{\partial x}(0, t) = 0 \tag{2b}$$

$$\frac{\partial u}{\partial x}(1, t) = -g[u(1, t), t]. \tag{2c}$$

Here $u(x, t)$ is the temperature of the system as a function of the time t and the spatial coordinate x. The dimensionless temperatures and the heat flux are defined in Table 1 of [1].